

# On the uniqueness for the 2D MHD equations without magnetic diffusion

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## Abstract

In this paper, we obtain the uniqueness of the 2D MHD equations, which fills the gap of recent work [2] by Chemin et al.

*Keywords:* MHD equations, Uniqueness, Magnetic diffusion

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## 1. Introduction

This paper considers the 2D MHD equations given by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0, \\ u(x, 0) = u_0(x), \quad B(x, 0) = B_0(x), \end{cases} \quad (1.1)$$

here  $t \geq 0$ ,  $x \in \mathbb{R}^2$ ,  $u = u(x, t)$  and  $B = B(x, t)$  are vector fields representing the velocity and the magnetic field, respectively,  $p = p(x, t)$  denotes the pressure and  $\nu$  is a positive viscosity constant.

(1.1) has been investigated by many mathematicians. In 2014, by establishing a generalized Kato-Ponce estimate (see [7] for the well-known result):

$$\langle u \cdot \nabla B \mid B \rangle_{\dot{H}^s} \leq C \|\nabla u\|_{H^s} \|B\|_{H^s}^2, \quad s > \frac{d}{2}, \quad d = 2, 3,$$

Fefferman et al. [4] obtained the local existence and uniqueness for (1.1) and related models with the initial data  $(u_0, B_0) \in H^s(\mathbb{R}^d)$ ,  $s > \frac{d}{2}$ . For other results concerning regularity criterions, we refer to [5] and [9].

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Very recently, Chemin et al. in [2] obtain the local existence for (1.1) in 2D and 3D. But for the 2D case, the uniqueness was not obtained. Our main result is filling the gap of their works. The details can be described as follows

**Theorem 1.1.** *For  $u_0 \in B_{2,1}^0(\mathbb{R}^2)$  and  $B_0 \in B_{2,1}^1(\mathbb{R}^2)$  with  $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ , there exists a time  $T = T(\nu, \|u_0\|_{B_{2,1}^0}, \|B\|_{B_{2,1}^1}) > 0$  such that the system (1.1) has a unique solution  $(u, B)$  with*

$$u \in C([0, T]; B_{2,1}^0(\mathbb{R}^2)) \cap L^1([0, T]; B_{2,1}^2)$$

and

$$B \in C([0, T]; B_{2,1}^1(\mathbb{R}^2)).$$

## 2. Preliminaries

Let  $\mathfrak{B} = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}$  and  $\mathfrak{C} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ . Choose two nonnegative smooth radial function  $\chi, \varphi$  supported, respectively, in  $\mathfrak{B}$  and  $\mathfrak{C}$  such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote  $\varphi_j = \varphi(2^{-j}\xi)$ ,  $h = \mathfrak{F}^{-1}\varphi$  and  $\tilde{h} = \mathfrak{F}^{-1}\chi$ , where  $\mathfrak{F}^{-1}$  stands for the inverse Fourier transform. Then the dyadic blocks  $\Delta_j$  and  $S_j$  can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) f(x - y) dy,$$

$$S_j f = \sum_{k \leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y) f(x - y) dy.$$

Formally,  $\Delta_j = S_j - S_{j-1}$  is a frequency projection to annulus  $\{\xi : C_1 2^j \leq |\xi| \leq C_2 2^j\}$ , and  $S_j$  is a frequency projection to the ball  $\{\xi : |\xi| \leq C 2^j\}$ . One can easily verifies that with our choice of  $\varphi$

$$\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \text{ and } \Delta_j (S_{k-1} f \Delta_k f) = 0 \text{ if } |j - k| \geq 5.$$

With the introduction of  $\Delta_j$  and  $S_j$ , let us recall the definition of the Besov space.

Let  $s \in \mathbb{R}$ ,  $(p, q) \in [1, \infty]^2$ , the homogeneous space  $\dot{B}_{p,q}^s$  is defined by

$$\dot{B}_{p,q}^s = \{f \in \mathfrak{S}'; \|f\|_{\dot{B}_{p,q}^s} < \infty\},$$

where

$$\|f\|_{\dot{B}_{p,q}^s} = \begin{cases} (\sum_{j \in \mathbb{Z}} 2^{sqj} \|\Delta_j f\|_{L^p}^q)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j f\|_{L^p}, & \text{for } q = \infty, \end{cases}$$

and  $\mathfrak{S}'$  denotes the dual space of  $\mathfrak{S} = \{f \in \mathcal{S}(\mathbb{R}^d); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index}\}$  and can be identified by the quotient space of  $\mathcal{S}'/\mathcal{P}$  with the polynomials space  $\mathcal{P}$ .

Let  $s > 0$ , and  $(p, q) \in [1, \infty]^2$ , the inhomogeneous Besov space  $B_{p,q}^s$  is defined by

$$B_{p,q}^s = \{f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{B_{p,q}^s} < \infty\},$$

where

$$\|f\|_{B_{p,q}^s} = \|f\|_{L^p} + \|f\|_{\dot{B}_{p,q}^s}.$$

Let's recall space-time space.

**Definition 2.1.** Let  $s \in \mathbb{R}$ .  $1 \leq p, q, r \leq \infty$ ,  $I \subset \mathbb{R}$  is an interval. The homogeneous mixed time-space Besov space  $\tilde{L}^r(I; \dot{B}_{p,q}^s)$  is defined as the set of all the distributions  $f$  satisfying

$$\|f\|_{\tilde{L}^r(I; \dot{B}_{p,q}^s)} = \left\| 2^{sj} \left( \int_I \|\Delta_j f(\tau)\|_{L^p}^r d\tau \right)^{\frac{1}{r}} \right\|_{l^q(\mathbb{Z})} < \infty.$$

For convenience, we sometimes use  $\tilde{L}_t^r \dot{B}_{p,q}^s$  and  $L_t^r \dot{B}_{p,q}^s$  to denote  $\tilde{L}^r(0, t; \dot{B}_{p,q}^s)$  and  $L^r(0, t; \dot{B}_{p,q}^s)$ .

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

**Proposition 2.2.** Let  $\alpha \geq 0$ . Let  $1 \leq p \leq q \leq \infty$ .

1) If  $f$  satisfies

$$\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K2^j\},$$

for some integer  $j$  and a constant  $K > 0$ , then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) If  $f$  satisfies

$$\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer  $j$  and constants  $0 < K_1 \leq K_2$ , then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where  $C_1$  and  $C_2$  are constants depending on  $\alpha, p$  and  $q$  only.

For more details about Besov space such as some useful embedding relations and the equivalency

$$\|f\|_{\dot{B}_{2,2}^s} \approx \|f\|_{\dot{H}^s}, \quad \|f\|_{B_{2,2}^s} \approx \|f\|_{H^s},$$

see [6],[1] and [8].

### 3. Proof of The main result

Before the proof of Theorem 1.1, we need the following lemma.

**Lemma 3.1.**  $\forall t > 0$ ,

$$\int_0^t \|f(\tau)\|_{\dot{B}_{2,1}^1} d\tau \leq C(\|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} + \|f\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}) \log \left( e + \frac{\|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^0} + \|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^2}}{\|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} + \|f\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}} \right). \quad (3.1)$$

*Proof.* Using the definition of homogeneous Besov space, we have

$$\begin{aligned} \|f\|_{L_t^1 \dot{B}_{2,1}^1} &= \sum_{j \in \mathbb{Z}} 2^j \|\Delta_j f\|_{L_t^1 L^2} \\ &= \sum_{j < -N} 2^j \|\Delta_j f\|_{L_t^1 L^2} + \sum_{-N \leq j \leq N} 2^j \|\Delta_j f\|_{L_t^1 L^2} + \sum_{j > N} 2^j \|\Delta_j f\|_{L_t^1 L^2} \\ &\leq 2^{-N} \|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^0} + 2N \|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} + 2^{-N} \|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^2}. \end{aligned}$$

Choosing

$$N = \log \left( e + \frac{\|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^0} + \|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^2}}{\|f\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} + \|f\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}} \right),$$

we can get the inequality (3.1) □

Now, we begin the proof of Theorem 1.1. The existence of the solution to (1.1) was obtained in [2], while the continuity in time can be obtained by the definition of Besov space. So here we only deal with the uniqueness. Let  $(u_j, B_j)$ ,  $j = 1, 2$ , be two solution of (1.1), denote  $\delta u = u_1 - u_2$ ,  $\delta B = B_1 - B_2$  and  $\delta p = p_1 - p_2$ , then we obtain

$$\partial_t \delta u + (u_1 \cdot \nabla) \delta u + (\delta u \cdot \nabla) u_2 - \nu \Delta \delta u + \nabla \delta p = (B_1 \cdot \nabla) \delta B + (\delta B \cdot \nabla) B_2 \quad (3.2)$$

and

$$\partial_t \delta B + (u_1 \cdot \nabla) \delta B + (\delta u \cdot \nabla) B_2 = (B_1 \cdot \nabla) \delta u + (\delta B \cdot \nabla) u_2. \quad (3.3)$$

First, we consider (3.2). By a standard argument, we have

$$\begin{aligned} \frac{d}{dt} \|\Delta_j \delta u\|_{L^2} + \nu 2^{2j} \|\Delta_j \delta u\|_{L^2} &\leq \|[\Delta_j, u_1 \cdot \nabla] \delta u\|_{L^2} \\ &\quad + \|\Delta_j(\delta u \cdot \nabla u_2)\|_{L^2} + \|\Delta_j(B_1 \cdot \nabla \delta B)\|_{L^2} + \|\Delta_j(\delta B \cdot \nabla B_2)\|_{L^2}, \end{aligned}$$

which with Gronwall's inequality yields that

$$\begin{aligned} \|\Delta_j \delta u\|_{L^2} &\leq \int_0^t e^{\nu 2^{2j}(\tau-t)} (\|[\Delta_j, u_1 \cdot \nabla] \delta u\|_{L^2} \\ &\quad + \|\Delta_j(\delta u \cdot \nabla u_2)\|_{L^2} + \|\Delta_j(B_1 \cdot \nabla \delta B)\|_{L^2} + \|\Delta_j(\delta B \cdot \nabla B_2)\|_{L^2}) d\tau. \end{aligned}$$

Taking the  $L^r(0, t)$  norm, and using Young's inequality to obtain

$$\begin{aligned} \|\Delta_j \delta u\|_{L_t^r L^2} &\leq \|e^{-\nu 2^{2j} \tau}\|_{L_t^r} (\|[\Delta_j, u_1 \cdot \nabla] \delta u\|_{L_t^1 L^2} \\ &\quad + \|\Delta_j(\delta u \cdot \nabla u_2)\|_{L_t^1 L^2} + \|\Delta_j(B_1 \cdot \nabla \delta B)\|_{L_t^1 L^2} + \|\Delta_j(\delta B \cdot \nabla B_2)\|_{L_t^1 L^2}) \\ &\leq (\nu 2^{2j} r)^{-\frac{1}{r}} (\|[\Delta_j, u_1 \cdot \nabla] \delta u\|_{L_t^1 L^2} \\ &\quad + \|\Delta_j(\delta u \cdot \nabla u_2)\|_{L_t^1 L^2} + \|\Delta_j(B_1 \cdot \nabla \delta B)\|_{L_t^1 L^2} + \|\Delta_j(\delta B \cdot \nabla B_2)\|_{L_t^1 L^2}). \end{aligned}$$

Multiplying  $2^{-j}$ , and taking the  $l^\infty$  norm, we obtain

$$\begin{aligned} \nu^{\frac{1}{r}} \|\delta u\|_{\tilde{L}_t^r \dot{B}_{2,\infty}^{-1+\frac{2}{r}}} &\leq \sup_{j \in \mathbb{Z}} 2^{-j} (\|[\Delta_j, u_1 \cdot \nabla] \delta u\|_{L_t^1 L^2} \\ &\quad + \|\Delta_j(\delta u \cdot \nabla u_2)\|_{L_t^1 L^2} + \|\Delta_j(B_1 \cdot \nabla \delta B)\|_{L_t^1 L^2} + \|\Delta_j(\delta B \cdot \nabla B_2)\|_{L_t^1 L^2}) \\ &= K_1 + K_2 + K_3 + K_4, \end{aligned}$$

where

$$K_1 = \sup_{j \in \mathbb{Z}} 2^{-j} \|[\Delta_j, u_1 \cdot \nabla] \delta u\|_{L_t^1 L^2}, \quad K_2 = \sup_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j(\delta u \cdot \nabla u_2)\|_{L_t^1 L^2},$$

$$K_3 = \sup_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j(B_1 \cdot \nabla \delta B)\|_{L_t^1 L^2}, \quad K_4 = \sup_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j(\delta B \cdot \nabla B_2)\|_{L_t^1 L^2}.$$

In the following, we will bound  $K_i$ ,  $i = 1, 2, 3, 4$ . By homogeneous Bony decomposition, we can split  $K_1$  into four parts,

$$\begin{aligned} K_1 &\leq \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{|k-j| \leq 4} \|[\Delta_j, S_{k-1} u_1 \cdot \nabla] \Delta_k \delta u\|_{L_t^1 L^2} + \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{|k-j| \leq 4} \|\Delta_j(\Delta_k u_1 \cdot \nabla S_{k-1} \delta u)\|_{L_t^1 L^2} \\ &\quad + \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \geq j-3} \|\Delta_k u_1 \cdot \nabla \Delta_j S_{k+1} \delta u\|_{L_t^1 L^2} + \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \geq j-3} \|\Delta_j(\Delta_k u_1 \cdot \nabla \tilde{\Delta}_k \delta u)\|_{L_t^1 L^2} \\ &= K_{11} + K_{12} + K_{13} + K_{14}, \end{aligned} \tag{3.4}$$

where  $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$ .

By Hölder's inequality, standard commutator estimate and Bernstein's inequality,

$$K_{11} \leq C \sup_{j \in \mathbb{Z}} 2^{-j} \|\nabla u_1\|_{L_t^1 L^\infty} \|\Delta_j \delta u\|_{L_t^\infty L^2} \leq C \|u_1\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}},$$

$$K_{12} \leq C \sup_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j u_1\|_{L_t^1 L^\infty} \|\nabla S_{j-1} \delta u\|_{L_t^\infty L^2} \leq C \|u_1\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}},$$

$$\begin{aligned} K_{13} &\leq C \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \geq j-3} 2^j \|\Delta_j \delta u\|_{L_t^\infty L^2} \|\Delta_k u_1\|_{L_t^1 L^\infty} \\ &\leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j-k} 2^{-j} \|\Delta_j \delta u\|_{L_t^\infty L^2} 2^k \|\Delta_k u_1\|_{L_t^1 L^\infty} \\ &\leq C \|u_1\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}} \end{aligned}$$

and

$$K_{14} \leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^j \|\Delta_k u_1\|_{L_t^1 L^2} \|\tilde{\Delta}_k \delta u\|_{L_t^\infty L^2} \leq C \|u_1\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}.$$

Collecting the estimates above in (3.4), we obtain

$$K_1 \leq C \|u_1\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}.$$

By homogeneous Bony decomposition again,

$$\begin{aligned} K_2 &\leq \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{|k-j| \leq 4} \|\Delta_j(\Delta_k \delta u \cdot \nabla S_{k-1} u_2)\|_{L_t^1 L^2} \\ &\quad + \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{|k-j| \leq 4} \|\Delta_j(S_{k-1} \delta u \cdot \nabla \Delta_k u_2)\|_{L_t^1 L^2} \end{aligned}$$

$$\begin{aligned}
& + \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{k \geq j-3} \|\Delta_j(\Delta_k \delta u \cdot \nabla \tilde{\Delta}_k u_2)\|_{L_t^1 L^2} \\
& = K_{21} + K_{22} + K_{23}.
\end{aligned}$$

By Hölder's inequality and Bernstein's inequality,

$$K_{21} \leq C \|\nabla u_2\|_{L_t^1 L^\infty} \sup_{j \in \mathbb{Z}} 2^{-j} \|\Delta_j \delta u\|_{L_t^\infty L^2} \leq C \|u_2\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}},$$

$$\begin{aligned}
K_{22} & \leq C \sup_{j \in \mathbb{Z}} 2^{-j} \|S_{j-1} \delta u\|_{L_t^\infty L^\infty} \|\nabla \Delta_j u_2\|_{L_t^1 L^2} \\
& \leq C \sup_{j \in \mathbb{Z}} 2^j \|\nabla \Delta_j u\|_{L_t^1 L^2} 2^{-j} \|S_{j-1} \delta u\|_{L_t^\infty L^2} \\
& \leq C \|u_2\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}},
\end{aligned}$$

$$\begin{aligned}
K_{23} & \leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^j \|\Delta_k \delta u\|_{L_t^\infty L^2} \|\tilde{\Delta}_k u_2\|_{L_t^1 L^2} \\
& \leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j-k} 2^{2k} \|\tilde{\Delta}_k u_2\|_{L_t^1 L^2} 2^{-k} \|\Delta_k \delta u\|_{L_t^\infty L^2} \\
& \leq C \|u_2\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}.
\end{aligned}$$

Thus we have

$$K_2 \leq C \|u_2\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}}.$$

Similarly, we can bound  $K_3$  and  $K_4$  as follows:

$$K_3 \leq \|B_1 \cdot \nabla \delta B\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^{-1}} \leq \int_0^t \|B_1 \cdot \nabla \delta B\|_{\dot{B}_{2,\infty}^{-1}} d\tau \leq \int_0^t \|B_1\|_{\dot{B}_{2,1}^1} \|\delta B\|_{\dot{B}_{2,\infty}^0} d\tau$$

and

$$K_4 \leq \|\delta B \cdot \nabla B_2\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^{-1}} \leq \int_0^t \|\delta B \cdot \nabla B_2\|_{\dot{B}_{2,\infty}^{-1}} d\tau \leq \int_0^t \|B_2\|_{\dot{B}_{2,1}^1} \|\delta B\|_{\dot{B}_{2,\infty}^0} d\tau.$$

Therefore,

$$\begin{aligned}
& \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}} + \nu \|\delta u\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} \\
& \leq C \|(u_1, u_2)\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}} + C \int_0^t \|(B_1, B_2)\|_{\dot{B}_{2,1}^1} \|\delta B\|_{\dot{B}_{2,\infty}^0} d\tau.
\end{aligned} \tag{3.5}$$

Next, we consider (3.3), we have the following estimate,

$$\begin{aligned}
\frac{d}{dt} \|\delta B\|_{\dot{B}_{2,\infty}^0} &\leq \sup_{j \in \mathbb{Z}} \|[\Delta_j, u_1 \cdot \nabla] \delta B\|_{L^2} \\
&\quad + \sup_{j \in \mathbb{Z}} \|\Delta_j(\delta u \cdot \nabla B_2)\|_{L^2} + \sup_{j \in \mathbb{Z}} \|\Delta_j(B_1 \cdot \nabla \delta u)\|_{L^2} + \sup_{j \in \mathbb{Z}} \|\Delta_j(\delta B \cdot \nabla u_2)\|_{L^2} \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By homogeneous Bony decomposition,

$$\begin{aligned}
J_1 &\leq \sup_{j \in \mathbb{Z}} \sum_{|k-j| \leq 4} \|[\Delta_j, S_{k-1} u_1 \cdot \nabla] \Delta_k \delta B\|_{L^2} + \sup_{j \in \mathbb{Z}} \sum_{|k-j| \leq 4} \|\Delta_j(\Delta_k u_1 \cdot \nabla S_{k-1} \delta B)\|_{L^2} \\
&\quad + \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} \|\Delta_k u_1 \cdot \nabla S_{k+1} \Delta_j \delta B\|_{L^2} + \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} \|\Delta_j(\Delta_k u_1 \cdot \nabla \tilde{\Delta}_k \delta B)\|_{L^2} \\
&= J_{11} + J_{12} + J_{13} + J_{14}.
\end{aligned}$$

By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned}
J_{11} &\leq C \sup_{j \in \mathbb{Z}} \|\nabla u_1\|_{L^\infty} \|\Delta_j \delta B\|_{L^2} \leq C \|u_1\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}, \\
J_{12} &\leq C \sup_{j \in \mathbb{Z}} 2^{2j} \|\Delta_j u_1\|_{L^2} 2^{-2j} \|\nabla S_{j-1} \delta B\|_{L^\infty} \leq C \|u_1\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}, \\
J_{13} &\leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^j \|\Delta_k u\|_{L^\infty} \|\Delta_j \delta B\|_{L^2} \\
&\leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j-k} 2^{2k} \|\Delta_k u\|_{L^2} \|\Delta_j \delta B\|_{L^2} \leq C \|u_1\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}, \\
J_{14} &\leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{2j} \|\Delta_k u\|_{L^2} \|\tilde{\Delta}_k \delta B\|_{L^2} \leq C \|u_1\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}.
\end{aligned}$$

Hence we have

$$J_1 \leq C \|u_1\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}.$$

By the inequality

$$\|fg\|_{\dot{B}_{2,1}^1} \leq C \|f\|_{\dot{B}_{2,1}^1} \|g\|_{\dot{B}_{2,1}^1},$$

(see, e.g., [3]), we have

$$J_2 + J_3 \leq \|\delta u \cdot \nabla B_2\|_{\dot{B}_{2,\infty}^0} + \|B_1 \cdot \nabla \delta u\|_{\dot{B}_{2,\infty}^0} \leq C \|\delta u\|_{\dot{B}_{2,1}^1} \|(B_1, B_2)\|_{\dot{B}_{2,1}^1}.$$



Finally, we bound  $J_4$ . By homogeneous Bony decomposition,

$$\begin{aligned}
J_4 &\leq \sup_{j \in \mathbb{Z}} \sum_{|k-j| \leq 4} \|\Delta_j(\Delta_k \delta B \cdot \nabla S_{k-1} u_2)\|_{L^2} + \sup_{j \in \mathbb{Z}} \sum_{|k-j| \leq 4} \|\Delta_j(S_{k-1} \delta B \cdot \nabla \Delta_k u_2)\|_{L^2} \\
&\quad + \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} \|\Delta_j(\Delta_k \delta B \cdot \nabla \tilde{\Delta}_k u_2)\|_{L^2} \\
&= J_{41} + J_{42} + J_{43},
\end{aligned}$$

where

$$\begin{aligned}
J_{41} &\leq C \sup_{j \in \mathbb{Z}} \|\nabla u_2\|_{L^\infty} \|\Delta_j \delta B\|_{L^2} \leq C \|u_2\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}, \\
J_{42} &\leq C \sup_{j \in \mathbb{Z}} 2^j \|\nabla \Delta_j u_2\|_{L^2} 2^{-j} \|S_{j-1} \delta B\|_{L^\infty} \leq C \|u_2\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}
\end{aligned}$$

and

$$J_{43} \leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{2j} \|\tilde{\Delta}_k u_2\|_{L^2} \|\Delta_k \delta B\|_{L^2} \leq C \|u_2\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}.$$

So

$$J_4 \leq C \|u_2\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0}.$$

Therefore,

$$\frac{d}{dt} \|\delta B\|_{\dot{B}_{2,\infty}^0} \leq C \|(u_1, u_2)\|_{\dot{B}_{2,1}^2} \|\delta B\|_{\dot{B}_{2,\infty}^0} + C \|(B_1, B_2)\|_{\dot{B}_{2,1}^1} \|\delta u\|_{\dot{B}_{2,1}^1},$$

which implies that  $\forall 0 \leq t \leq T$ ,  $T$  is the lifespan of the solution,

$$\|\delta B\|_{L_t^\infty \dot{B}_{2,\infty}^0} \leq C \|(u_1, u_2)\|_{L_t^1 \dot{B}_{2,1}^2} \|\delta B\|_{L_t^\infty \dot{B}_{2,\infty}^0} + C \|(B_1, B_2)\|_{L_t^\infty \dot{B}_{2,1}^1} \|\delta u\|_{L_t^1 \dot{B}_{2,1}^1}. \quad (3.6)$$

Set  $0 < \bar{T} < T$  such that

$$\int_0^{\bar{T}} \|(u_1, u_2)\|_{\dot{B}_{2,1}^2} dt < \frac{1}{4C},$$

then  $\forall 0 \leq t \leq \bar{T}$ , (3.5) and (3.6) reduce to

$$\frac{3}{4} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}} + \nu \|\delta u\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} \leq \int_0^t \|(B_1, B_2)\|_{\dot{B}_{2,1}^1} \|\delta B\|_{\dot{B}_{2,\infty}^0} d\tau \quad (3.7)$$

and

$$\frac{3}{4} \|\delta B\|_{L_t^\infty \dot{B}_{2,\infty}^0} \leq C \|(B_1, B_2)\|_{L_t^\infty \dot{B}_{2,1}^1} \|\delta u\|_{L_t^1 \dot{B}_{2,1}^1}. \quad (3.8)$$

Plugging (3.8) into (3.7) yields that

$$\frac{3}{4} \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}} + \nu \|\delta u\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1} \leq C \int_0^t \|(B_1, B_2)(\tau)\|_{\dot{B}_{2,1}^1} \|(B_1, B_2)\|_{L_\tau^\infty \dot{B}_{2,1}^1} \|\delta u\|_{L_\tau^1 \dot{B}_{2,1}^1} d\tau$$

$$\leq C_T \int_0^t \|\delta u\|_{L_\tau^1 \dot{B}_{2,1}^1} d\tau.$$

Thanks to the Log-type inequality (3.1), denote

$$X(t) = \|\delta u\|_{L_t^\infty \dot{B}_{2,\infty}^{-1}} + \|\delta u\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^1},$$

we have

$$X(t) \leq C_{T,\nu} \int_0^t X(\tau) \log \left( e + \frac{V(\tau)}{X(\tau)} \right) d\tau,$$

where  $V(t) = \|\delta u\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^0} + \|\delta u\|_{\tilde{L}_t^1 \dot{B}_{2,\infty}^2}$ , is bounded in  $[0, T]$ . Applying the Osgood's Lemma (see, [1] p.125) and combining with (3.6) can yields  $\delta u = \delta B = 0$  in  $[0, \bar{T}]$ . By a standard continuous argument, we can show that  $\delta u = \delta B = 0$  in  $[0, T] \times \mathbb{R}^2$ . This completes the proof of Theorem 1.1.

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